Section Four

Money Management

The importance of money management and bet sizing has been stressed increasingly in recent years and rightly so. For even if the player has discovered a favorable betting situation, how he wagers determines his success or failure. Ultimately, it is the "bottom line" on which a gambler's performance is judged. It is fine, of course, to describe the favorable situation to a friend or business associate, but the next question is likely to be "How much money are you making from this situation?"

The problem for the gambler is that much of the advice on money management is conflicting or confusing, or simply based on false premises. There are hundreds of schemes designed to overcome the house edge in roulette and craps based solely on manipulating the size of one's bets. As will be seen, all such attempts are futile.

Even assuming the player has discovered a favorable game (i.e., one offering a positive expectation), the question naturally arises: How does one best use a limited amount of capital to exploit this positive expectation? Wager too boldly and the player risks losing his entire bankroll, even though he or she may have made only favorable bets. This is commonly known as gambler's ruin. On the other hand, betting too conservatively the player severely limits his opportunity to make a good return on his capital.

Fortunately for the player, there exists a mathematical theory for committing resources in favorable games. This will be discussed in Chapter 9.

Chapter 8

Mathematical Systems

Before looking at the optimal strategy for exploiting a positive expectancy situation, it may be worthwhile to evaluate what I refer to as mathematical systems. Although here I use roulette as an example, the principles apply equally to craps and the Wheel of Fortune.

By a "mathematical system" I mean a system where the player decides which bet to make using only the following information:

(1) a record of what numbers have come up on some number of past spins, and

(2) a record of the bets he has made, if any, on those spins.

We assume here that when the player bets, for him all numbers are equally likely to come up on each spin of the wheel. This means not using biased wheels or physical prediction method.

Roulette has long been the prototype of unbeatable gambling games. It is normally regarded as a repeated independent trials process which generates at each trial precisely one from a set of random numbers. Players may wager on particular subsets of random numbers (e.g., the first dozen, even, an individual number, etc.), winning if the number which comes up is a number of the chosen subset. A player may wager on several subsets simultaneously and each bet is settled without references to the others. To fix the discussion, let's consider the standard American wheel. This has thirty-eight numbers, namely 0, 00, 1, 2,. . .36.

The mathematician's *assumption*, that each of these numbers is equally likely beforehand to come up on any spin of the ball and wheel, seems plausible. The wheels are carefully machined and balanced by the manufacturer. They are checked from time to time by the casinos. When they show signs of wear they may be thoroughly reconditioned. Even if the wheel has irregularities which make some numbers more favored than others, if the player does not know this and his system is not designed to exploit this, then mathematical reasoning—based on the assumption that all numbers are equally likely to come up—gives correct conclusions about that player's system.

The Doubling-up System

One more assumption must be made to properly evaluate mathematical systems. We must also assume there is a smallest allowable house (minimum) bet (such as $1) and a greatest allowable house (maximum) bet (such as $1000). Casinos need to fix a maximum bet in order to stop the simple mathematical system of "doubling up." To see why, imagine we've found a casino with no maximum. We bet $1000, because Red pays even money or 1 for 1. If we lose, we double and bet $2000 on the second turn. If that wins, we net $1000 on the two turns. If the second bet loses, we double again and bet $4000 on the third turn. Having lost $3000 on the first two turns, a win of $4000 on the third turn nets $1000 on the cycle of three turns. We continue doubling our bet after each loss. Finally, when we win, we have a net gain of $1000. We put this $1000 safely aside and start a new cycle of doubling until we win with a bet of $1000 on the Red. Each completed cycle wins another $1000 net. Table 8-1 illustrates this cycle.

The doubling-up system in Table 8-1 with no casino limit on bets is being discussed *not* because anyone would ever be allowed

Table 8-1

turn #	amount bet	total profit if cycle ends on this turn	chance cycle ends on or before this turn: exact	chance cycle ends on or before this turn: decimal approximation
1	$ 1,000	$1,000	$1-(20/38)$	0.4737
2	$ 2,000	$1,000	$1-(20/38)^2$	0.7230
3	$ 4,000	"	$1-(20/38)^3$	0.8542
4	$ 8,000	"	$1-(20/38)^4$	0.9233
5	$ 16,000	"	$1-(20/38)^5$	0.9596
6	$ 32,000	"	$1-(20/38)^6$	0.9787
7	$ 64,000	"	$1-(20/38)^7$	0.9888
8	$ 128,000	"	$1-(20/38)^8$	0.9941
9	$ 256,000	"	$1-(20/38)^9$	0.9969
10	$ 512,000	"	$1-(20/38)^{10}$	0.9984
11	$1,024,000	"	$1-(20/38)^{11}$	0.9991
31	$\$1000 \times 2^{30}$ or about a trillion	"	$1-(20/38)^{31}$	0.999,999,997,7
36	$\$1000 \times 2^{35}$ or about 34 trillion	"	$1-(20/38)^{36}$	0.999,999,999,9
100	about $\$6 \times 10^{32}$	"	$1-(20/38)^{100}$	
n	$\$1000 \times 2^{n-1}$	$1,000	$1-(20/38)^n$	

to do it, but to illustrate ideas we will be using. To see how ridiculous the system would be, note that if the first ten turns of a cycle have lost, on the eleventh turn the player bets 1,024 times his initial bet. His initial bet was $1,000, so he bets $1,024,000. Of course the chance is small that this will happen. The last column shows a chance of 0.9984 that the cycle ends on or before the tenth turn, hence that the eleventh bet is never made. Thus, the chance of reaching the eleventh turn is only 1 − 0.9984 = 0.0016 or 0.16% or about one chance in 613. But if the doubling-up system is used long enough, it will happen.

With 30 losses in a row, the player is supposed to bet about one trillion dollars on the thirty-first turn. This is about the net worth of the New York Stock Exchange. On turn 36, the bet is about $34 trillion. This exceeds the net worth of the world! (The net worth of the U.S.A. is about 6 trillion current dollars. I'd guess the net worth of the world to be about $30 trillion.) The player should arrange from the start to have unlimited credit, *reasonably* pointing out that since he must eventually win he is sure to pay off!

Real casinos don't go for this. They have house limits (which they may increase sometimes under special circumstances) and credit limits. So this "sure-fire winning system" is never used. But players for centuries have used modified doubling-up systems in actual casino play. An illustration is given in Table 8-2. Here the player starts by betting $1 on Red. He keeps doubling his bet until he wins. Then he starts the cycle over with a $1 bet on Red. Each cycle produces a $1 profit *unless*—and here is the catch—he loses ten times in a row and then wants to bet $1024 on the eleventh turn of the cycle. The house limit prevents that and prevents further doubling if the player loses on his eleventh turn.

Notice from Table 8–2 that if the player wins after nine or fewer losses, he wins $1 and successfully completes the cycle. But if he loses ten times in a row, he can bet only $1000 on the eleventh turn. If he then wins, he loses "only" $23 on this cycle. But if he loses on the eleventh turn, he loses $2023 on the cycle, for a major disaster. Of course, the chance of ever reaching the eleventh turn of a cycle is as we saw before, only about one chance in 613.

Is this system any good, or do the chances of loss on the eleventh turn ruin it?

We are going to find out that the "house percentage advantage" on Red is not changed *in the slightest* by the doubling-up system. In fact, the disaster of the eleventh turn is *exact* compensation to the casino for the high chance the player has of winning $1 per cycle. We will show this by a computation. But what is perhaps truly amazing is that this is also true for all mathematical systems, no matter how complex, including all those that can ever

Table 8-2

turn #	amount bet	total $ losses before bet	net profit if cycle ends, this turn	chance of this result: exact	chance of this result: decimal approximation
1	1	0	1	$18/38$	0.4737
2	2	1	1	$20/38 \times 18/38$	0.2493
3	4	3	1	$(20/38)^2 \times 18/38$	0.1312
4	8	7	1	$(20/38)^3 \times 18/38$	0.0691
5	16	15	1	$(20/38)^4 \times 18/38$	0.0363
6	32	31	1	$(20/38)^5 \times 18/38$	0.0191
7	64	63	1	$(20/38)^6 \times 18/38$	0.0101
8	128	127	1	$(20/38)^7 \times 18/38$	0.0053
9	256	255	1	$(20/38)^8 \times 18/38$	0.002789
10	512	511	1	$(20/38)^9 \times 18/38$	0.001468
11	1000	1023	-23	$(20/38)^{10} \times 18/38$	0.000773
			or -2023	$(20/38)^{11}$	0.000858
				total = 1	

be discovered. Since there are an infinite number of such systems, we cannot prove this by computation (an infinite amount of time would be needed to do the required infinite number of computations). Instead, I will indicate how the mathematician, by logic (like the logic of, say, plane geometry with its axioms, theorems and proofs) can show that none of this infinite number of systems is any good.

A lot of what I'm saying is easier than it sounds. For instance, to see that there are an infinite number of systems for roulette, all, I have to do is give you any endless list of systems. Here is one such list (always bet on Red); System 1. Bet $1 on Red if Red came up one turn ago; if it didn't come up one turn ago, bet $2. System 2. Always bet $1 on Red if it came up two turns ago; if it did not come up two turns ago, bet $2. And so on for systems 3, 4, . . . etc.

I didn't say my list of systems would be interesting, only that it would be endless!

The doubling-up system can be good for some fun even if it doesn't alter the house edge. Suppose you're in Las Vegas with your spouse or your date. It's almost dinner time and you say casually, "Dinner for two will run us about thirty dollars. Why don't we eat for free? I'll just pick up $30 at this roulette wheel. It'll only take a few minutes." If you have $2100 in your pocket and the house limits are from $1 to $1000 on Red, you can use the doubling-up system. You need to complete 30 cycles without ever having a string of eleven losses. You will win $1 per cycle, for a total of $30, and be off to dinner.

How safe is this scheme? What are your chances? Table 8-1 says that the chance a cycle lasts 10 turns or less, and therefore you win $1, is 0.9984. The chance that you do this 30 times in a row turns out to be 0.9984^{30} or 0.9522, so the chance you will succeed is over 95%. If you set your sights lower, say $20 or $10, then the chances of success go up to 96.79% and 98.38%, respectively. But be warned: if you fail, you can lose as much as $2023.

An important factor in determining the risk of failure is the ratio of the house maximum bet on Red to the minimum bet. To illustrate, suppose instead of $1 to $1000 for a ratio of 1000, the betting limits were $2 to $500, for a ratio of 500/2 = 250. Then if we start a cycle with a $2 bet, we hit the house limit on the ninth spin, after eight losses. (To see this, use Table 8-2 and double all the numbers in the second, third and fourth columns, because we *start* with a $2 bet rather than a $1 bet, as before.) Now the chance the cycle ends in eight turns or less is (from the last column of Table 8-1) 0.9941. Thus to win $30 you need to complete 15 cycles, the chance of which is 0.9941^{15} or 0.9152. If you try this in a roulette game with better odds, say single-zero European style, the chance of success increases.

The doubling-up system is one of a class of systems that are sometimes called martingales. The origin of the term is given in the American Heritage Dictionary, New College Edition, which is the most informative definition I have seen on this. The word

evolved from a similarly named village of Martigues in the Provence district of southern France, whose residents were viewed as peculiar and were roundly ridiculed with Gallic expertise. Their bizarre behavior included such things as gambling with the doubling-up system and lacing up their pants from behind. To use the doubling-up system became known as gambling "al la martigalo" (fem), "in the Martigues manner," i.e., "in a ridiculous manner."

There are many other popular "mathematical" systems. "Tripling up," where the player bets 1,3,9,27, etc. until he wins, then repeats, is like doubling up, but it wins faster and runs into trouble (in the form of the house limit) faster.

If you want to know more about "mathematical systems," consider these books:

The book *Casino Gambling, Why You Win, Why You Lose,* by Russell T. Barnhart (Brandywine, N.Y., 1978, $12.95). Barnhart is a skilled magician and a longtime student of gambling. He has gambled extensively all over the world so he knows both the theory and practice of his subject. The book has 50,000 spins from an actual wheel and an elaborate discussion of mathematical or "staking" systems.

Allan Wilson's classic *Casino Gambler's Guide* has considerable material on systems and their fallacies. His treatment of biased roulette wheels may be the best ever written.

Richard Epstein's engaging treatise, *The Theory of Gambling and Statistical Logic, Revised,* (Academic Press, 1977) is a landmark in the subject. Much of it requires a university-level mathematics background. However, it is the best single reference work in print on the general subject of games and gambling, and even the general reader can glean much from browsing through it.

Now I'll explain why mathematical systems like the doubling-up system, cannot reduce the casino percentage.

The Problem with Doubling Up

One reason I chose roulette to illustrate mathematical systems is that it is easy to understand the odds and probabilities.

One correct version of the so-called "law of averages" says that in a "long" series of bets, you will *tend* to gain or lose "about" the total expectation of those bets. This means that a series of "bad" bets is also "bad," and that systems don't help.

Applying these ideas to the doubling-up system, let's calculate the player's expectation for one *cycle*. Think of a complete cycle as a single (complicated-looking) bet. Now refer to Table 8-2. The fifth column gives the probability that the cycle ends on turn #1, #2, etc. and the fourth column gives the gain or loss for each of these cases. Multiply each entry in the fourth column by the corresponding entry in the fifth column. Then add the results:

$\$1 \times 18/38 + \$1 \times 20/38 \times 18/38 + \ldots + \$1 \times (20/38)^9 \times 18/38 - \$23 \times (20/38)^{10} \times 18/38 - \$23 \times (20/38)^{10} \times 18/38 - \$2023 \times (20/38)^{11}$ which simplifies to $1 - 24 \times (20/38)^{10} - 2000 \times (20/38)^{11} = 1 - 0.0391\ldots - 1.7168\ldots = -\$.7560266578\ldots\ldots$ Thus, the expected loss to the bettor is about $-\$.76$ per cycle.

Now let's calculate the expected (or "average") amount bet on one cycle. Referring again to Table 8-2, we see that if the cycle ends on turn #1, the total of all bets is $1, if it ends on turn #2, the total of all bets is $1 + $2, if it ends on turn #3, the total is $1 + $2 + $4, etc. If the cycle ends on turn #11, the total amount bet is $2,023. (To get these totals as of the end of any turn, add columns two and three.) Then multiply these total amounts bet by the chances in column five to get $\$1 \times 18/38 + \$2 \times (20/38) \times (18/38) + \$4 \times (20/38)^2 \times (18/38) + \ldots + \$512 \times (20/38)^9 \times (18/38) + \$2023 \times (20/38)^{10} \times (18/38) + \$2023 \times (20/38)^{11}$ which simplifies to $\$2 \times (18/38) \times ((40/38)^{10} - 1)/(40/38 - 1) + \$2024 \times (20/38)^{10} - \$1 = \$14.3645065$. If we divide the expected loss by the average bet per cycle we get $-\$.756\ldots \div \$14.36\ldots 1/19$ exactly or -5.26%.

These calculations are tedious, and for each system the details are different, so they have to be done again. And there are an infinite number of gambling systems, so calculations can never check them all out anyhow. Clearly this is not the way to understand gambling systems. The correct way is to develop a general

mathematical theory to cover gambling systems. That has been done and here's how it works. First we define the *action* in a specified set of bets to be the total of all bets made. From what we have said, your expected (gain or) loss is your action (i.e., the total of all your bets) times the house edge. For example, if you bet $10 per hand at blackjack and play for 10 hours, betting 100 hands per hour, you have made a thousand $10 bets, which is $10,000 worth of "action." If you are a poor blackjack player and the casino has a 3% edge over you, your expected loss is $\$10{,}000 \times 3\% = \$300$. Your actual loss may be somewhat more or somewhat less.

If Nevada casino blackjack grosses a total of $400 million per year and the average casino edge over the player is 2% of the initial wager, then we can determine the total action (A) per year: $.02A = \$400{,}000{,}000$ so $A = \$20$ billion. Thus from these figures we would estimate $20 billion worth of bets are made per year at Nevada blackjack. The 2% figure might be substantially off. We could get a fairly accurate idea of the true figure by making a careful statistical sampling survey. If, instead, the figure is 4%, then $A = \$10$ billion. With 1%, $A = \$40$ billion per year.

Guidelines for Evaluating Systems

The general principles we have discussed apply to almost all gambling games, and when they apply, they guarantee that systems cannot give the player an advantage.

To help you reject systems, here are conditions which guarantee that a system is worthless:

I. Each individual bet in the game has negative expectation. (This makes *any series* of bets have negative expectation.)

II. There is a maximum limit to the size of any possible game. (This rules out systems like the no-limit doubling up system discussed.)

III. The results of any one play of the game do not "influence" the results of any other play of the game. (Thus, in roulette, we assume that the chances are equally likely for all of the numbers on each and every future spin, regardless of the results of past spins.)

IV. There is a minimum allowed size for any bet. (This is necessary for the technical steps in the mathematical proof. Most people would take for granted that there is such a minimum, namely some multiple of the smallest monetary unit. In the U.S.A., the minimum allowed bet is some multiple of one cent. In West Germany, it may be some multiple of the pfenning, and so forth.)

Under these conditions, it is a mathematical fact that every possible gambling system is worthless in the following ways:

(1) Any series of bets has negative expectation.

(2) This expectation is the (negative) sum of the expectations of the individual bets.

(3) If the player continues to bet, his total loss divided by his total action will tend to get closer and closer to his expected loss divided by his total action.

(4) If the player continues to bet it is almost certain that he will:

(a) be a loser;

(b) eventually stay a loser forever, and so never again break even;

(c) eventually lose his entire bankroll, no matter how large it was.

To give you an idea of how valuable this result is for spotting worthless systems, here are some examples of systems which cannot possibly give the player an advantage:

1. All the roulette systems I have ever heard of, except the following two types. (a) Biased wheels, in which condition (I) may be violated; the numbers are no longer equally likely, so bets on some numbers may have positive expectation. (b) Physical prediction methods, in which the position and velocity of ball and rotor are used to predict the outcome.

2. All craps systems I have ever heard of, except possibly those using either crooked dice or physical "control" of dice.

(Note: While at the Fifth Annual Gambling Conference at Lake

Tahoe, I saw a dice cheat control the dice, at a private showing. I then saw him win at a casino. I heard he did this regularly. His badly mutilated body was found in the Las Vegas area a year later.

3. Any systems for playing keno, slots and chuck-a-luck.

As a further illustration, consider the book *Gambling Systems That WIN*, published by Gambling Times, 1978, paperback, $2. Of the fourteen systems given there, our result applies at once to eight. (The other six are one blackjack system, four racing systems, and a basketball system.)

(In the case of sports bets, it is generally difficult to determine whether condition I is satisfied. In the case of blackjack, condition I fails if the player counts cards, and there are, in fact, some winning systems, as most of you know.)

This leaves eight systems in *WIN:* four craps systems, one baccarat system, two roulette systems, and a keno system.

Conditions I through IV hold for all eight systems so none of them are winning systems. Nor do any of them reduce the house edge in the slightest. However, they may be entertaining. Also, in games like keno, craps, and roulette, where the expectation may vary from one game to another or from one type of bet to another, some ways to bet are "smarter" (translation—less dumb; more accurate translation—less negative expectation but still losing) than others.

For those who are prepared to lose, but want to lose more slowly, such systems may be of interest.

In most cases, the basic information is a list of the various bets in the game and their expectation. Then, if you must play, choose only bets with the least negative expectation. The "system" complexities and hieroglyphics are not essential.

It may amuse you to see why condition IV is needed. Suppose, instead, that there is *no* minimum bet and that we are playing Red at roulette. Our first bet is $1,000. There is an 18/38 chance that we win $1,000 and a 20/38 chance we lose $1,000. Now suppose that the second bet is $0.90, the third bet is $0.09, the fourth bet is $0.009, the fifth bet is $0.0009, etc. (Remember: *no* minimum.) Then the total of all bets from the second on is $0.99999...=$1.00.

The total gain or loss on these bets is between—$1.00 and +$1.00. The total action on all bets is $1,000+$1=$1,001.

If we won the first bet, our total winnings (T) will always be between $999 and $1,001. This happens with probability 18/38. Therefore, conclusions 4(a), 4(b), and 4(c) fail. Also, our total action is $1,001 so T/A is always between $999/$1,001 and $1,001/$1,001. But our expected gain (E) is negative so E/A is less than 0. Therefore, if we win the first bet, T/A does not tend to get closer and closer to E/A. Therefore, conclusion 3 also fails.

Conclusion 4(c) also deserves some comment. Actually, there is an insignificantly small chance the player can win the casino's bankroll before losing his. But even for moderate-size casino bankrolls,this possibility is so tiny as to be negligible, no matter how large the player's bankroll! We will discuss this in the next chapter. It is also discussed at some length in the 1962 edition of my book *Beat the Dealer,* and in Feller's great *An Introduction to Probability and its Applications, Vol. I,* Wiley. Thus, a more exact version of conditions I-IV would include information about the size of the casino bankroll. Then conclusion 4 would include information about the tiny chance that 4(a), (b), and (c) don't happen.

As far as I know, the most elementary mathematical proof ever given for all this is in my textbook, *Elementary Probability,* available from Robert E. Krieger Publishing Co., Inc., 645 New York Avenue, Huntington, New York 11743. The proof is outlined on pp. 84-85, exercises 5.12 and 5.13. It requires no calculus and can be followed by a good high school mathematics student if he or she works through pp. 1-85.

We now have a powerful test for showing that a system doesn't win. This keeps us from wasting our money and time buying or playing losing systems. It also helps us in our search for systems that do win, by greatly narrowing the possibilities.

Chapter 9

Optimal Betting

It is somewhat ridiculous to discuss an optimal money management strategy when the player has a negative expectancy. As indicated in Chapter 8, with an enforced house maximum and minimum wager, there is no way to convert a negative expectation into a positive expectation through money manipulation. Any good money management plan says not to wager in such a situation. Players facing a negative expectancy should look elsewhere for a gambling game or, at the very least, bet insignificant amounts and write off in their mind the expected loss as "entertainment."

After the gambler has discovered a favorable wagering situation, he is faced with the problem of how best to apportion his limited financial resources. There exists a rule or formula which you can use to decide how much to bet. I will explain the rule and tell you the benefits that are likely if you follow it.

Let's begin with a simple illustration that I deliberately exaggerated to better get the idea across. Suppose you have a very rich adversary who will let you bet any amount on heads at each toss of a coin and that you both know that the chance of heads is some number p greater than ½. If your bet pays even money, then you have an edge. Now suppose $p = 0.52$, so you tend to win 52 percent of your bets and lose 48 percent. This is similar to the situation in blackjack when the ten-count ratio is about 1.5 percent. Suppose too that your bankroll is only $200. How much should you bet? You could play safe and just bet one cent each time. That way, you would have virtually no chance of ever losing your $200 and being put out of the game. But your expected gain is .04 per unit or .04 cents per bet. At 100 one cent bets an hour, you expect to win four cents per hour. It's hardly worth playing.

Now look at the other extreme where you bet your whole bankroll. Your expected gain is $4 on the first bet, more than if you bet any lesser amount. If you win, you now have $200. If you again bet all of it on your second turn, your expected gain is $8 and is more than if you bet any lesser amount. You make your expected gain the biggest on each turn by betting everything. But if you lose once, you are broke and out of the game. After many turns, say 20, you have won 20 straight tosses with probability. $.52^{20} = 0.000002090$ and have a fortune of $104,857,600, or you have lost once with probability 0.999997910 and have nothing. In general, as the number of tosses increases, the probability that you will be ruined tends to 1 or certainty. This makes the strategy of betting everything unattractive.

Since the gambling probabilities and payoffs at each bet are the same, it seems reasonable to expect that the "best" strategy will always involve betting the same fraction of your bankroll at each turn. But what fraction should this be? The "answer" is to bet $p - (1 - p) = 0.52 - 0.48 = 0.04$, or four percent of your bankroll each time. Thus you bet $4 the first time. If you win, you have $104, so you bet 0.04 × $104 = $4.16 on the second turn. If you lost the first turn, you have $96, so you bet $0.04 \times \$96 = \$3.84$ on the second turn. You continue to bet four percent of your bankroll at each turn. This strategy of "investing" four percent of your bankroll at each trial and holding the remainder in cash is known in investment circles as the "optimal geometric growth

portfolio" or OGGP. In the 1962 edition of *Beat the Dealer,* I discussed its application to blackjack at some length. There I called it the Kelly system, after one of the mathematicians who studied it, and I also referred to it as (optimal) fixed fraction (of your bankroll) betting.

Why is the Kelly system good? First, the chance of ruin is "small." In fact, if money were infinitely divisible (which it can be if we use bookkeeping instead of actual coins and bills, or if we use precious metals such a gold or silver), then any system where you never bet everything will have zero chance of ruin because even if you always lose, you still have something left after each bet. The Kelly system has this feature. Of course, in actual practice coins, bills or chips are generally used, and there is a minimum size bet. Therefore, with a very unlucky series of bets, one could eventually have so little left that he has to bet more of his bankroll than the system calls for. For instance, if the minimum bet were $1, then in our coin example, you must overbet once your bankroll is below $25. If the minimum bet were one cent, then you only have to overbet once your bankroll falls below 25 cents. If the bad luck then continues, you could be wiped out.

The second desirable property of the Kelly system is that if someone with a significantly different money management system bets on the same game, your total bankroll will probably grow faster than his. In fact, as the game continues indefinitely, your bankroll will tend to exceed his by any preassigned multiple.

The third desirable property of the Kelly system is that you tend to reach a specified level of winnings in the least average time. For example, suppose you are a winning card counter at blackjack, and you want to run your $400 bankroll up to $40,000. The number of hands you'll have to play on average to do this will, using the Kelly system, be very close to the minimum possible using any system of money management.

To summarize, the Kelly system is relatively safe, you tend to have more profit, and you tend to get to your goal in the shortest time.

Blackjack Money Management

The Kelly system calls for no bet unless you have the advantage. Therefore, it would tell you to avoid games such as craps and keno and slot machines. However, if you have the knowledge and skill to gain an edge in blackjack, you can use the Kelly system to optimize your rate of gain. The situation in blackjack is more complex than the coin toss game because (1) the payoff on a one-unit initial bet can vary widely, due to such things as dealer or player blackjacks, insurance, doubling down, pair splitting, and surrender, and (2) because the advantage or disadvantage to the player varies from hand to hand.

However, we can apply the coin toss results to blackjack by making some slight modifications. First, let's see where the coin toss example's best fixed fraction of four percent came from. The general mathematical formula for the Kelly system is this: In any (single) favorable gambling situation or investment, bet that fraction of your bankroll which maximizes $E \ln (1 + f)$, where E is the expected value and $\ln$ is the natural logarithm (to the base $e = 2.71828\ldots$). This $\ln$ function is available on most hand calculators. In the case of coin tossing, the best fraction, which I call f^*, is given for a favorable bet by $f^* = 2p - 1$, where p is the chance of success on one toss, and $f^* = 0$ if $p = 1/2$, i.e., if the game is either fair or to your disadvantage. Note too that $f^* = 2p - 1$ is coincidentally your expected gain per unit bet.

Now your expected gain in blackjack varies from hand to hand. If we think of successive hands as coin tosses with a varying p, then we should bet $f^* = 2p - 1$ whenever our card count shows that the deck is favorable. When the deck is unfavorable, we "should" bet zero. Uston-type team play approximates this ideal of betting zero in unfavorable situations. You can also achieve this sometimes by counting the deck and waiting until the deck is favorable before placing your first bet. But it is impractical to bet zero in unfavorable situations, so we bet as small as is discreet. Think of these smaller, slightly unfavorable bets as a "drain" or "tax" which "water down" the overall advantage of the

favorable bets. To compensate for this reduced advantage, f^* should generally be "slightly" smaller than the $2p - 1$ computed above. Another effect of the small, slightly unfavorable bets is to increase the chance of ruin a little.

The most important blackjack "correction" to the f^* computed for coin tossing is due to the greater variability of payoff. Peter Griffin calculates that the "root mean square" payoff on a one-unit blackjack bet is about 1.13. It turns out then that f^* should be corrected to about $(2p - 1)/1.27$ or about .79 times the advantage. Shade this to .75 because of the "drain" of the small, unfavorable bets and we have the farily accurate rule: For favorable situations at blackjack, it is (Kelly) optimal to bet a percent of your bankroll equal to about 3/4 percent advantage. For instance, with a $400 bankroll and a one percent advantage, bet 3/4 of one percent of $400, or $3.

The Kelly System for Roulette

In general in roulette, the house has the edge, and the Kelly system says, "don't bet." But in my chapter on physical prediction at roulette, I described a method where we (Shannon and I), with the aid of an electronic device, had an edge of approximately 44 percent on the most favored single number. That corresponds to a win probability of $p = 0.04$, with a payoff of 35 times the bet, and a probability of $1 - p = 0.96$ of losing the bet. It turns out that $f^* = .44/35 = .01257$. The general formula for f^* when you win A times a favorable bet with probability p and lose the bet with probability $1 - p$, is $f^* = e/A$ where $e = (A + 1)p - 1 > 0$ the player's expected gain per unit bet or his advantage. Here $A = 35$, $p = .04$, and $e = 0.44$. In the coin toss example, $A = 1$, $p = .52$, and $e = .04$.

Using any fixed betting function f, the "growth rate" of your fortune is $G(f) = p \ln (1 + Af) + (1 - p) \ln(1 - f)$. After N bets you will have approximately $\exp[N \quad G(f)]$ times as much money, where $\exp$ is the exponential function, also given on most pocket calculators.

For the roulette single number example, using my hand calculator (an HP65) gives $G(f^*)=0.04 \ln(1+35f^*)+0.96 \ln(1-f^*)=.04 \ln(1.44)+0.96 \ln(0.98743)=.04 \times .36464+0.96 \times (-0.01265)=0.1459-.01215=.00244$. After 1,000 bets, you will have approximately $exp[2.44]=11.47$ times your starting bankroll.

Notice the small value of f^*. That's because the very high risk of loss on each bet makes it too dangerous to bet a large fraction of your bankroll. To show the advantages of diversification, suppose instead that we divide our bet equally among the five most favored numbers, as Shannon and I actually did in the casinos. If one of these numbers come up, we win an amount equal to $(35-4)/5$ of our amount bet, and if none come up, we lose our bet. Thus $A=31/5=6.2$ The other four numbers are not quite as favored as the best number. However, to illustrate diversification, suppose that the five-way bet has the same .44 advantage. This corresponds to $p=0.20$. Then $f^*=.44/6.2=0.07097$, so you bet about seven percent of your bankroll and $G(f^*)=0.20 \ln(1+6.2f^*)+0.80 \ln(1-f^*)=0.01404$. This growth rate is about 5.75 times that for the single number. After 1,000 bets, you would have approximately 1.25 million times your starting bankroll. Such is the power of diversification.

What is the price of deviating from betting the optimal Kelly fraction f^*? It turns out that for bet payoffs like blackjack, which can be approximated by coin tossing, the "performance loss" is not serious over several days play. But for the roulette example, the performance loss from moderate deviations from the Kelly system is considerable.

APPENDICES

APPENDIX A.

Suppose point count systems which are "closer" to the relative u_i values of Table 2-2 are likely to be "better." To test this we require a precise meaning for "better" and a precise measure of "closeness." We begin by basing the definition of "better" on the notions of probabilistic dominance, and of risk, used in mathematical finance.

Definition 1. Let F and G be probability distribution functions. Then F *probabilistically dominates* G if $F(x) \leq G(x)$ for all x. If in addition $F(x_0) < G(x_0)$ for at least one x_0 then F *strictly probabilistically dominates* G. If F and G arise from random variables X and Y, respectively, or from probability measures *u* and *v*, respectively, then the defined terms apply to these pairs if they hold for F and G.

That F probabilistically dominates G is equivalent to $P(X \geq x) \geq P(Y \geq x)$ for all x. If X is the player expectation from point count system A and Y is the player expectation from system B, then this means that the chance of finding expectations of x or more is always at least as good as using A as it is by using B. One can show that this means that a player following A has at least as great an expected return as B with "the same risk level."

However, probabilistic dominance is inadequate as a definition of "better" because the typical situation is that F is "spread out" more in both directions from the mean full deck expectation $E_0=0$. Thus F dominates G for $x > E_0$ and G dominates F for $x < E_0$. In fact G is (to a good approximation) a convex contraction of F. More precisely, if E_F and E_G are the respective means of F and G, we will find $E_F \geq E_G \geq E_0$ with Y-E_G a convex contraction (this is equivalent to the notion "less risky than" of portfolio theory); of X - E_F. Thus F is both "spread out more" than G and translated in the positive direction more. The reason why E_F, $E_G \geq E_0$ is because E_0 is the expectation using the basic strategy and constant bets, equivalent to the full pack expectation. When (advantageous) counting systems are used, the strategy for playing hands is improved whenever the player has seen any cards other than the ones he and the dealer use on the first round. Since this generally happens with positive probability, we then have E_F, $E_G > E_0$.

Definition 2. Point count system A is *better than* system B if E_F E_G and also $P(X \geq x)$ $P(Y \geq x)$ for $x \geq E_G$.

Typically count systems satisfy $E_F \geq E_G \geq E_0$ and X-E_F=a(Y-E_G), $a \geq 1$ (a special case of convex contraction). These conditions imply A is better than B.

Assume that the betting systems b(E) are numerical functions of the expectation E. Further assume $b(E)=1$ if $E \leq 0$ and $b(E) \geq 1$ if $E > 0$. These are the ones generally considered. The popular fallacious systems such as the martingales (e.g. "doubling up"), and the La Bouchere which incorporate past results, are of no interest here.

Theorem 3. With the preceding notation and assumptions, if A is *better than* B, then for any betting system $b_B(E)$ based on the B point count, there is a betting system $b_A(E)$ based on the A point count such that the return R_A per unit bet by A (approximately) probabilistically dominates R_B. Further, R_A and R_B have approximately the same risk. In fact $R_A=R_B+c$, where $c \geq 0$.

Proof: If F and G are continuous, define b_A by $b_A(F^{-1}(G(E)))=b_B(E)$. Then note that the first unit of each bet has expectation E_A for A and E_B for B. The remainder of the bet is non-zero only if $E \geq E_F$. Then for corresponding percentiles of the respective distributions, A places the sames bets as B. But $F(E) \leq G(E)$ if $E \geq E_F$ so A has in each instance at least as great expectation, hence has at least as great expectation overall. Thus the total expected return to A is at least as large as for B. Also $R_A \geq R_B$ per unit since the bets placed have the same distribution.

In reality F and G are not continuous; instead they are finite. But they may be arbitrarily closely approximated by continuous distributions so the result extends, with one qualification. If F or G is discontinuous, extend the graphs of F and G by adding vertical segments at the discontinuity points so that the extensions $\tilde{F}$ and $\tilde{G}$ have inverses defined on (0,1). Then for those E′ such that G is discontinuous at E′ or F is discontinuous at $F^{-1}(G(E'))$ it may be necessary to define $b_A(F^{-1}(G(E')))$ "probabilistically", so it is multiple-valued, each value occurring with specified probabilities.

To show that $R_A=R_B+c$, which implies the same risk, it suffices to assume that at each percentile level y for the distributions F and G we have the conditional distributions given y satisfying $F(x|y)=G(x\text{-}f(y)|y)$ where $f(y) \geq 0$. Since this only holds approximately in practice, we have $R_A=R_B+c$.

Now we turn to the problem of measuring "closeness" of a given count to the "ultimate" strategy. We shall assume that point count strategies are of the form $C=(c_1,c_2,\ldots c_{13})$ where c_1 is the value assigned for an ace, $c_2,\ldots,c_9$ are the point counts for ranks 2 through 9, and $c_{10}=\ldots=c_{13}$ are the point counts for tens, jacks, queens, and kings respectively. In practice these are lumped together and only ten point count values are specified. By writing C with 13 components we gain a symmetry which yields substantially simpler proofs. Note that C and aC, a=0, are equivalent and will be identified.

Definition 4. If $\Sigma\Delta E_i=0$ the *ultimate strategy* $U=(u_1,\ldots u_{13})$ is the one given by $u_i=\Delta E_i$ where ΔE_i is the change in expectation from removing one ith card from the complete pack. If $d=\Sigma\Delta E_i=0$ then U is given by u_i-d/13.

In Table 2-2, we have d for one deck is .024 and d for four decks is .017. The u_i rows are calculated in Table 2-2 from Definition 4.

It is tempting to think of U as representing to good approximation the direction of the gradient E at $f_1=\ldots=f_{13}=1/13$ of the player's expectation $E(f_1,\ldots,f_{13})$ as a function of the fraction f_i of the cards from i=1 to 13. Then we calculate (C)=C• U/‖C‖•U/‖C‖•‖U‖, i.e. the projection of C in the E direction. The numerator is the inner or scalar product and $\|C\|=(\Sigma c_i^2)^{1/2}$.

Next we claim that λ (C) gives the approximate ratio of the spread of the C distribution F_c about E_c to the U distribution F_u about E_u. Then λ (C) is the desired measure of closeness. In particular, for approximately the same risk per unit, and the same distribution of the bet sizes, it would follow that $E(R_u) \doteq E(R_c)/(C)$. Then C_1 and C_2 are arbitrary strategies $E(R_{c1})/E\ (R_{c2}$ $\doteq \lambda\ (C_1)/\ \lambda\ (C_2)$ for the same risk level and distribution of bet sizes. Thus the "power" of a strategy C is proportional to its λ (C).

This conclusion is true but the argument must resolve two obstacles:

(1) In the preceding discussion we treated C, U, ∇E, etc. as though they were given in Cartesian coordinates when in fact they are not.

(2) The probability distribution of $E(f_1,\ldots,f_{13}$ must be considered in reaching the conclusion and in general will invalidate it.

Note further that both U and C are linear approximations to an in general curved "surface". Also in the real case the domain is a large finite subset of points of the possible $(f_1,\ldots,f_{13})$, each of positive probability. (The original discovery of winning blackjack systems [Thorp, 1961], was motiviated by this model.) First I introduced the $E(n,\ldots,n_{13})$ "surface", where n_i is the number of cards remaining of denomination i. Intuitive arguments "con-

vinced" me that the E surface should have substantial deviations from E_0, the full deck expectation. The next step was to approximate by "the" $E(f_1,\ldots,f_{13})$ "surface", and then to "linearize" the problem by assuming that $E(f_1,\ldots,f_n) \doteq E_0 + \Sigma_{ki}\ \Delta\ f_i$, where $\Delta\ f_i=f_i$-1/13.) Thus there is the approximation of a discrete problem by a continuous one. Nonetheless, we shall show:

Theorem 5. If the probability distribution of $(f_1,\ldots,f_{13})$ is approximately rotationally symmetric about (1,...,1)/13 then the relative power of any point count system C is proportional to (C)=C.U/‖C‖•‖U‖. The powers of two count systems which exploit the count information equally (e.g. if one normalized by the number of as yet unseen cards so does the other; if one carries a side ace count for betting and sets the ace equal to 0 for strategy, so does the other, etc.) are approximately proportional to their λ's.

Proof.

APPENDIX B.

Suppose (Hypothesis I) that the shoe really has four complete decks. Then the number X of unseen ten-value cards among the 104 cards (two decks) not seen will average 32. In the general case with U unseen cards, T tens in the whole pack, and N non-tens in the whole pack, the average value A of X is given by $A = UT/(N + T)$. In our example, $U = 104$, $T = 64$ and $N = 144$, so we get $A = 104 \times 64/208 = 32$. But there will be a fluctuation around this number. Mathematicians use the standard deviation S to measure this fluctuation. The formula $S^2 = [UTN/T + N)^2](1 - (U - 1)/N + T - 1)]$.

For our example, $S^2 = (104 \times 64 \times 144/208^2)[(1 - 103/207)(= 11.1304$, so $S = \sqrt{11.1304} = 3.3362$. To a good approximation, X is "normally distributed" with mean $A = 32$ and standard deviation $S = 3.3362$.

Now, suppose instead (Hypothesis II) that the deck has ten ten-value cards removed. Then $U = 94$, $T = 54$ and $N = 134$. If Y is the number of unseen cards, we have the real $A = 25.6364$, but we think there are ten more ten-value cards. So assuming incorrectly that no ten-values are gone, the number that we deduce for Y has an average of $A + 10 = 35.6364$. The real S^2 for Y is $94 \times 54 \times 134/198^2\ (1 - 93/197) = \sqrt{9.1593}$, so $S = 3.0264$.

What we want to know is whether to believe Hypothesis I ("null hypothesis") or Hypothesis II. This is a classic statistics problem. It turns out that in order for us to have a good chance to believe the correct hypothesis, the A value for X and Y need to be at least two and preferably several S units apart. In this example, they differ by only $35.6364 - 32 = 3.6364$ which is about one S unit. Of course, repeated countdowns of this same shoe will again increase our ability to tell whether the shoe is short.

APPENDIX C.

For this first simple discussion, let's suppose $\underline{x}(t) = a \exp(bt) + c$, where a, b, and c are constants and *exp* is the exponential function. This is one of the simplest mathematical functions that has the right "shape." (Note: Mathematical readers may wish to redo this discussion using the quadratic $\underline{x}(t) = at^2 + bt + c$ to see the difference.)

I recall that the ball velocity at the point where it fell from the track was about 0.5 revolutions per second (r.p.s.) and that ten revolutions earlier it was about 2 r.p.s. Using this and the choice $t = 0$ when the ball leaves the track gives $a = 10/3$, $b = 3/20$, and $c = -10/3$. Thus, $x(t) = 10(\exp(3t/20) - 1)/3$ in r.p.s., and this gives an angular velocity v in r.p.s. of $v(t) = \frac{1}{2}\exp(3t/20)$. Figure 4-1 shows a graph of $x(t)$.

APPENDIX D.

A calculation shows, for our illustrative $x(t)$ function, that $x_0(T) = 1/\exp(3T/20) - 1) - 10/3$. Thus, from T we can predict the number of revolutions until the ball leaves the track. For instance, if $T = 1$ *sec.*, we predict the ball will leave the track in $x_0(1) = 1/(\exp(3/20) - 1) - 10/3 = 2.85$ revolutions after the switch is hit the second time. If instead $T = \frac{1}{2}$ *sec.*, then we predict $x_0(\frac{1}{2}) = 9.51$ revolutions.

APPENDIX E.

Math readers: $dx_0(T)/dT = -(3x_0(T) + 10)^2/60$. It can be shown that for the $x(t)$ of this example, the error $\Delta x_0 T$ in the prediction of $x_0(T)$ due to an error ΔT in measuring T, is given by $\Delta x_0(T) = -(3x_0(T) + 10)^2 T/60 = -3\ T/(20[exp(3T/20) - 1]^2)$. For instance, if $T = 0.8$ sec. and $\Delta T = 0.012$ sec., we have a prediction error of $\Delta x_0(0.8) = 0.11$ revs or 4.2 numbers on the wheel. In our illustration $T = 0.8$ sec. means $x_0(T) = 4.51$ revolutions to go. The time to go is $(20/3)log_e(3x_0(t)/10 + 1)$ or 5.70 sec. We have somewhat less time than this to bet.

APPENDIX F.

In our example, the equation for $t_0(T)$ is $t_0(T) = (20/3)log_e(3/10)/exp(3T/20) - 1)\ ) = (20/3)log_e(3x_0(T)/10 + 1)$. The error is approximately $\Delta t_0(T) = -(\Delta T)exp(3T/20) /(exp(3T/20) - 1)$. Thus again, if $T = 0.8$ sec. and $\Delta T = 0.012$ sec., $\Delta t_0(T) = -0.106$ sec. With a rotor speed of 0.33 r.p.s., this causes a rotor prediction error of 0.036 rev. or 1.3 pockets. In our example then, we measured T too large by 0.012 sec. This led us to believe the ball would leave the track at a point about 4.2 pockets before where it did. Therefore, we forecast impact on the rotor 4.2 pockets early. It also led us to believe the ball would leave the track sooner in time. Thus, we thought the rotor wouldn't revolve as far as it did. This made us forecast impact another 1.3 pockets early, for a total error of 5.5 pockets early. There are other important sources of error, so our final predictions were not this good. But they were good enough.

In summary, note that an error where ΔT is positive, i.e., we think T is bigger than it really is because we hit the switch early the first time or late the second time, leads us to think the ball

is slower than it is. That makes us think $x_0(T)$ is shorter. Thus, we expect the ball at the rotor too soon and forecast impact on the rotor ahead of where it tends to occur. Conversely, if T is negative (last on the first switch or early on the second), we think T is smaller, the ball is faster, and mistakenly forecast $x_0(T)$ and $t_0(T)$ as too big. Then we predict impact behind where it tends to occur.

The rotor angular velocity, followed a law close to $r(t) = Aexp(-bt)$. A typical value for A was 0.33 rev./sec. The "decay" or "slowing down" constant b was very small. The rotor is massive and spins on a well-oiled bearing (on our casino wheel, it was the pointed end of a sturdy steel shaft). In the course of a minute or two, the slowing was hardly perceptible. (Note: Stroboscopic "beat frequency" techniques, plus an accurate clock, can quickly and easily give a very precise measurement of b and the slowing down.)

Let's take $b = -log_e(10/11)/120$ or 0.000794/sec., which corresponds to a slowing down from 0.33 rev./sec. to 0.30 rev./sec. in two minutes. This seems like the right order of magnitude. To put the rotor position into the tiny computer we were going to build, we planned to hit a rotor timing switch once when the zero passed a reference mark on the wheel, and then hit the switch again when the zero passed the reference mark a second time. Since the rotor velocity was small and nearly constant, this was a less "sensitive" measurement. Therefore, we planned to do it first, shortly before the ball was spun.

How much error in the ball's final position (pocket) comes from rotor timing errors? Assume for simplicity that the rotor makes one revolution in about three seconds (.33 rev./sec.) and that we can neglect the slowing down of the rotor. Then, as in the ball timing, we might expect a typical (root mean square) size of about 11.2/1,000 seconds for the combined effect of the two errors. If the rotor really makes one revolution in 3.000 seconds, and we think it takes 3.0112 seconds, then in 30 seconds we think the wheel will travel 9.9628 revolutions whereas it really travels 10.0000 revolutions. Thus, the rotor goes .0372 rev. or 1.4 pockets farther than expected. Similarly, if we think the rotor takes 2.9888 seconds for one revolution, then in 30 seconds the rotor goes .0375 rev. or 1.4 pockets less than we expected.

APPENDIX G.

I am using the normal approximation for the statistical discussion. I think it is very nearly an accurate description of what happens and that this approximation only slightly affects the discussion.

APPENDIX H.

In general, there are exactly (5+r)!/5!r! home board positions with exactly r men. There are exactly (6+r)!/6!r!-1 home board positions with from one to r men. Thus, since r=15 is possible in the actual game, there are a total of 21!/6! 15! -1=54,263 different home board positions for one player. The symbol r!, read "r factorial," means 1×2×3×. . .xr. Thus 1!=1, 2!=2, 3!=6, 4! =24, etc.

Scholarly References

For those readers who are especially interested in the technical work behind the material in this book and other work by Professor Thorp, here is a list of some of his related scholarly publications.

Game Theory

1. "A Favorable Strategy for Twenty-One," *Proceedings of the National Academy of Sciences 47* (1961). 110-112. (MR #B846).

2. BOOK: *Beat the Dealer: A Winning Strategy for the Game of Twenty-One,* Random House, New York, 1962; Vintage paperback also, 1966; revised edition, 1966.

3. "A Partial Analysis Of Go," (with W. E. Walden), *The (British) Computer Journal* 7:3 (1964), 203-207. (MR33 #2424), (Z *123*, 340).

4. "A Favorable Side Bet in Nevada Baccarat," (with W. E. Walden), *Journal of the American Statistical Association 61,* Part I (1966), 313-328.

5. "Repeated Independent Trials and a Class of Dice Problems," (Mathematical Note), *American Mathematical Monthly* (August-September 1964), 778-781. (Z 123, 364).

6. "Optimal Gambling Systems for Favorable Games," *Review of International Institute of Statistics* 37:3 (1969), 273-293. (Z *191*, 497).

7. "Solution of a Poker Variant," *Information Sciences* 2/2 (1970), 299-301. (MR S42 #8952), (Z *205*, 232).

8. "A Computer-Assisted Study of Go on M x N Boards," (with W. E. Walden), Lecture Notes in Operations Research and Mathematical Systems in *Theoretical Approaches to Non-Numerical Problem Solving*, Vol. 28, R. Banerji and M. D. Mesarovic, eds., Springer-Verlag, New York, 1970, 303-343; later revised version, *Information Sciences* 4:1 (1972), 1-33 (MR45 #8684), (Z 228.68028).

9. "Non-Random Shuffling With Applications to the Game of Faro," *Journal of the American Statistical Association*, 842-847, December 1973. Much expanded version appears in *Gambling and Society*, edited by W. Eadington, Charles C. Thomas, Springfield, Illinois, 1975, as: Probabilities and Strategies for the Game of Faro, pp. 531-560.

10. "The Fundamental Theorem of Card Counting With Applications to Trente et Quarante and Baccarat," (with W. E. Walden) *International Journal of Game Theory* 2 (1973), 109-119. (Z 258.90054)

11. "Backgammon: Part I, The Optimal Strategy for the Pure Running Game," Proceedings of the Second Annual Conference on Gambling, Lake Tahoe, 1975. ms. 42 + pp. See news report on this paper in: "Beating the Game," an article by Dietrick E. Thomsen, *Science News*, Vol. 107, March, 1975.

12. "Blackjack Systems," Proceedings of the Second Annual Conferrence on Gambling, Lake Tahoe, 1975. ms. 15 + pp. See news report on this paper in: "Beating the Game," an article by Dietrick E. Thomsen, *Science News*, Vol. 107, March, 1975.

Probability, Statistics

13. BOOK: *Elementary Probability*, Wiley, New York, 1966.

Mathematical Finance

14. BOOK: *Beat the Market*, (with S. Kassouf), Random House, New York, 1967.

15. "Portfolio Choice and the Kelly Criterion," *Proceedings of the 1971 Business and Economics Section of the American Statistical Association* 1972, 215-224. To be reprinted in *Investment Decision-Making*, edited by J. Bicksler. Reprinted in *Stochastic Optimization Models in Finance*, Academic Press, edited by W. T. Ziemba, S. L. Brumelle, and R. G. Vickson, 1975, pp. 599-620.

16. "The Capital Growth Model: An Empirical Investigation," (with James Bicksler), *Journal of Financial and Quantitative Analysis*, March 1973, Vol. VIII, No. 2, pp. 273-287.

General Interest

17. "A Professor Beats the Gamblers", *The Atlantic Monthly*, June, 1962. Reprinted in "The Gambler's Bedside Book," John K. Hutchens, ed., Tapliniger, New York, 1977, pp. 166-177.